ON THE NUMBER OF LATIN RECTANGLES

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I. Symbolic Representation of Sieve Process

1.—Let Ω be a finite set and 2^Ω the set of all subsets of Ω. We denote the empty subset by $\emptyset$. Consider the free module (free abelian group) M over 2^Ω. If we define the product xy of two basic elements x and y of M by

$$xy = x \cap y \qquad \text{for } x, y \in 2^\Omega , \tag{1}$$

then we can regard M as a *ring*, where the multiplication is an extension of the relations (1). We call an element of M a *symbolic subset* of Ω.

2.—Now we define a binary operation $\vee$ in the ring M by

$$f \vee g = f + g - fg \qquad \text{for } f, g \in M .$$

It is easy to see that this operation is commutative and associative. More precisely, we have, by an induction, that

$$\begin{aligned} &f_1 \vee f_2 \vee \cdots \vee f_n \\ &\quad = \sum_i f_i - \sum_{i<j} f_i f_j + \sum_{i<j<k} f_i f_j f_k - + \cdots + (-1)^{n-1} f_1 f_2 \cdots f_n , \end{aligned}$$

for $f_1, f_2, \cdots, f_n \in M$.

3.—Suppose that μ is an *abstract measure* defined over 2^Ω, namely that μ is a complex-valued function defined over 2^Ω satisfying the conditions

$$\mu(\emptyset) = 0 , \tag{2}$$

and

$$\mu(x \cup y) = \mu(x) + \mu(y) - \mu(x \cap y) \qquad \text{for } x, y \in 2^\Omega . \tag{3}$$

Since the module M is generated by 2^Ω, we can extend μ to an *additive function* defined over M, which we shall denote by the same symbol μ. Thus (3) is now written as

$$\mu(x \cup y) = \mu(x \vee y) \qquad \text{for } x, y \in 2^\Omega ,$$

and, we have in general

$$\mu(x_1 \cup x_2 \cup \cdots \cup x_n) = \mu(x_1 \vee x_2 \vee \cdots \vee x_n) \qquad \text{for } x_1, x_2, \cdots, x_n \in 2^\Omega .$$

The above is a symbolic representation of the *sieve process* or the *principle of inclusion and exclusion*. The meaning of the above interpretation is that *in order to find the measure* $\mu(x_1 \cup x_2 \cup \cdots \cup x_n)$ *of an actual subset* $x_1 \cup x_2 \cup \cdots \cup x_n$ *of* Ω, *we need first develop the symbolic subset* $x_1 \vee x_2 \vee \cdots \vee x_n$ *in a linear combination of the basic elements, or actual basic subsets, of M, and then operate the additive function* μ.

4.—For an element x of 2^Ω, denote its complement in Ω by x'. Then we have from (2) and (3) that

$$\mu(x) + \mu(x') = \mu(\Omega) ,$$

or

$$\mu(x') = \mu(\Omega - x) .$$

Thus, when we are dealing with abstract measures μ, we may identify x' with the symbolic subset $\Omega - x$. The sieve process described above in § 3 states that

$$\mu\Big(\bigcap_{i=1}^{n} x_i'\Big) = \mu\Big(\prod_{i=1}^{n} (\Omega - x_i)\Big) ,$$

namely the measure of the *actual subset* $\bigcap_{i=1}^{n} x_i'$ is obtained first by expanding the *symbolic subset*

$$\prod_{i=1}^{n} (\Omega - x_i) = \Omega - x_1 \vee x_2 \vee \cdots \vee x_n$$

in terms of basic elements and then operating the *additive function* μ.

5.—Of special importance for application is the case where $\mu(x) = \# x$ is the number of elements in the subset x of Ω. Thus the number of elements belonging to none of the n subsets $x_1, x_2, \cdots, x_n$ of Ω is given by $\# y_0$, where y_0 is the symbolic subset

$$y_0 = \prod_{i=1}^{n} (\Omega - x_i) = \sum_{r=0}^{n} (-1)^r \sum_{I_r} x(I_r) , \tag{4}$$

where the inner summation is over all r-subsets (subsets consisting of r elements) I_r of $\{1, 2, \cdots, n\}$, and in general $x(I)$ for a subset I of $\{1, 2, \cdots, n\}$ is defined by

$$\begin{aligned} x(\emptyset) &= \Omega \qquad (\emptyset \text{ is the empty subset of } \{1, 2, \cdots, n\}) , \\ x(I) &= \prod_{i \in I} x_i \qquad (I \neq \emptyset) . \end{aligned} \tag{5}$$

Similarly the number of elements of Ω belonging to exactly k of the subsets $x_1, x_2, \cdots, x_n$ is given by $\# y_k$, where y_k is the symbolic subset

$$y_k = \sum_{r=k}^{n} (-1)^{r-k} \binom{r}{k} \sum_{I_r} x(I_r) .$$

6.—The number of elements of Ω belonging to at least k of the n subsets $x_1, x_2, \cdots, x_n$ is expressed as $\# z_k$, where z_k is the symbolic subset

$$z_0 = \Omega \,,$$

$$z_k = \sum_{r=k}^{n} (-1)^{r-k} \binom{r-1}{k-1} \sum_{I_r} x(I_r) \qquad \text{for } k \geqq 1 \,.$$

The number of elements of Ω belonging to at most k of the subsets $x_1, x_2, \cdots, x_n$ is expressed as $\#(\Omega - z_{k+1})$,

$$\Omega - z_{k+1} = \Omega + \sum_{r=k+1}^{n} (-1)^{r-k} \binom{r-1}{k} \sum_{I_r} x(I_r) \,.$$

Therefore we have the following identity in M.

$$\prod_{I_k} (\Omega - x(I_k)) = \Omega + \sum_{r=k}^{n} (-1)^{r-k-1} \binom{r-1}{k-1} \sum_{I_r} x(I_r) \qquad \text{for } k \geqq 1 \,.$$

In fact the left-hand side member corresponds to the actual subset consisting of elements of Ω belonging to at most $k-1$ of the subsets $x_1, x_2, \cdots, x_n$. The special case $k=1$ is nothing but the original sieve process (4) itself. In the sequel we shall make use of the case $k=2$ in the form

$$\prod_{I_2} (\Omega - x(I_2)) = \sum_{r=0}^{n} (-1)^r (1-r) \sum_{I_r} x(I_r) \,. \qquad (6)$$

II. The Number of Permutations Discordant With a Given Latin Rectangle

7.—Let L be a $k \times n$ Latin rectangle in the integers $1, 2, \cdots, n$. We denote the set of integers contained in the i-th column of L by $C(i)$ for $i = 1, 2, \cdots, n$. In this section we are mainly concerned with the number N of permutations of $1, 2, \cdots, n$ *discordant* with L, namely with the number of permutations σ such that $\sigma(i) \notin C(i)$ for $i = 1, 2, \cdots, n$. This number N depends only on the indexed subset $\{C(i); i = 1, \cdots, n\}$, and is equal to the number of *systems of distinct representatives* (SDR) of the indexed subsets $\{C'(i); i = 1, \cdots, n\}$, where $C'(i)$ is the complement in $\{1, \cdots, n\}$ of $C(i)$.

8.—Let $\Omega = \mathfrak{S}_n$ denote the set of all permutations of $1, \cdots, n$, and by x_{ij} the set of all permutations σ such that $\sigma(i) = j$. Then the condition that σ is discordant with L is that $\sigma \notin x_{ij}$ for all $j \in C(i)$, for $i = 1, \cdots, n$. Thus we have $N = \# w$, where w is the symbolic subset

$$w = \prod_{i=1}^{n} \prod_{j \in C(i)} (\Omega - x_{ij}) .$$

Expanding the above we have

$$w = \sum_{r=0}^{kn} (-1)^r \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \sum_{j_1 \in C(i_1)} \cdots \sum_{j_r \in C(i_r)} x_{i_1 j_1} \cdots x_{i_r j_r} .$$

Here we notice that $x_{i_1 j_1} x_{i_2 j_2} = \emptyset$ if $i_1 = i_2, j_1 \neq j_2$ or $i_1 \neq i_2, j_1 = j_2$. Thus $x_{i_1 j_1} \cdots x_{i_r j_r} = \emptyset$ unless all the i's are distinct and all the j's are distinct, namely unless $i_1, \cdots, i_r$ are all distinct and $\{j_1, \cdots, j_r\}$ is an SDR of the indexed set $\{C(i_1), \cdots, C(i_r)\}$. And if this condition is satisfied, then $\# x_{i_1 j_1} \cdots x_{i_r j_r} = (n-r)!$. So we have that

$$N = \sum_{r=0}^{n} (-1)^r \alpha_r (n-r)! , \tag{7}$$

$$\alpha_r = \sum_{I_r} D(I_r) , \tag{8}$$

where $D(I_r)$ is the number of SDR's of the indexed set $\{C(i); i \in I_r\}$, and the summation is over all the r-subsets I_r of $\{1, \cdots, n\}$.

9.—For a fixed r-subset I_r we find $D(I_r)$ by the sieve process. Now, let Ω be the set of all r-sequences $a = (a_i)_{i \in I}$, with $a_i \in C(i)$ for $i \in I_r$. Denote by $x_{j,i}$ the subset of Ω consisting of $a = (a_i)_{i \in I}$, $a_i \in C(i)$ for $i \in I_r$ such that $a_i = j$. Similarly to (5) of § 5 we define

$$x_j(I) = \prod_{i \in I} x_{j,i}$$

for any subset I of $\{1, \cdots, n\}$. Then the condition that a is an SDR is that a does not belong to any subset $x_j(P)$ for a 2-subset P of I_r. Thus we have

$$D(I_r) = \# z(I_r) ,$$

$$z(I_r) = \prod_{j=1}^{n} \prod_{P} (\Omega - x_j(P)) ,$$

where the inner product is taken over all 2-subsets P of I_r. This inner product, for a fixed j, represents a symbolic subset which was already dealt with in § 6, (6). Thus we have

$$z(I_r) = \prod_{j=1}^{n} \prod_{P} (\Omega - x_j(P)) = \prod_{j=1}^{n} \prod_{s_j=0}^{r} (-1)^{s_j}(1 - s_j) \sum_{Q_{s_j}} x_j(Q_{s_j}),$$

where the summation $\sum_{Q_{s_j}} x_j(Q_{s_j})$ is taken over all the s_j-subsets Q_{s_j} of I_r. Therefore we have that the above is equal to

$$\sum_{s_1=0}^{r} \cdots \sum_{s_n=0}^{r} (-1)^{s_1 + \cdots + s_n} (1-s_1) \cdots (1-s_n) \sum_{Q_{s_1}} \cdots \sum_{Q_{s_n}} x_1(Q_{s_1}) \cdots x_n(Q_{s_n}) .$$

10.—Here we notice that if $Q_{s_i} Q_{s_j} \neq \emptyset$, $i \neq j$ then $x_i(Q_{s_i}) x_j(Q_{s_j}) =$

$\emptyset$, since $m \in Q_{s_i} Q_{s_j}$, $a = (a_i)_{i \in I} \in x_i(Q_{s_i}) x_j(Q_{s_j})$ requires $a_m = i = j$. On the other hand $x_j(Q) = \emptyset$ unless $j \in C(i)$ for all i in Q by the definition of $x_j(Q)$. If, $Q_{s_1}, \cdots, Q_{s_n}$ are mutually disjoint and satisfy the condition that $j \in C(i)$ for all i in Q_{s_j}, for all j, then we have

$$\# x_1(Q_{s_1}) \cdots x_n(Q_{s_n}) = k^{r-(s_1+\cdots+s_n)} ,$$

since we can choose each of the $r - (s_1 + \cdots + s_n)$ components a_i of a with $i \notin Q_{s_1} \cup \cdots \cup Q_{s_n}$ quite arbitrarily in $C(i)$.

11.—Twisted around slightly we can describe the above symbolic subset $z(I_r)$ as follows. Let $s_1, \cdots, s_n$ be any integers between 0 and n, and choose an s_i-subset Q_{s_i} of $\{1, \cdots, n\}$ for $i = 1, \cdots, n$ to satisfy the conditions that

(*1) $Q_{s_1}, \cdots, Q_{s_n}$ are mutually disjoint, and that

(*2) $j \in C(i)$ for all i in Q_{s_j}, for $j = 1, \cdots, n$.

(Notice that the condition (*2) implies that $s_i \leqq k$ for all i.) Then, the symbolic subset $z(I_r)$ is the sum of

$$(-1)^{s_1+\cdots+s_n}(1 - s_1) \cdots (1 - s_n) x_1(Q_{s_1}) \cdots x_n(Q_{s_n})$$

for all choices as above, under one further condition that

(*3) $Q_{s_i} \in I_r$ for $i = 1, \cdots, n$.

Thus we have that

$$D(I_r) = \# z(I_r)$$
$$= \sum_{s_1=1}^{k} \cdots \sum_{s_n=0}^{k} \sum^{*}_{Q_{s_1} \subset I_r, \cdots, Q_{s_n} \subset I_r} (-1)^{s_1+\cdots+s_n}(1 - s_1) \cdots (1 - s_n) k^{r-(s_1+\cdots+s_n)} ,$$

where the sum $\sum^*$ is over all s_1-subsets $Q_{s_1}, \cdots,$ all s_n-subsets Q_{s_n} of $\{1, \cdots, n\}$ satisfying the conditions (*1) and (*2) above and (*3).

12.—Summing up $D(I_r)$ above over all r-subsets I_r of $\{1, \cdots, n\}$ we have that

$$\alpha_r = \sum_{s_1=0}^{k} \cdots \sum_{s_n=0}^{k} \sum^{*}_{Q_{s_1}, \cdots, Q_{s_n}} (-1)^{s_1+\cdots+s_n}(1 - s_1) \cdots (1 - s_n) \binom{n - (s_1 + \cdots + s_n)}{r - (s_1 + \cdots + s_n)} k^{r-(s_1+\cdots+s_n)} , \quad (9)$$

where the summation $\sum^*$ is over all s_1-subsets $Q_{s_1}, \cdots,$ all s_n-subsets Q_{s_n} of $\{1, \cdots, n\}$, satisfying the conditions (*1) and (*2), since for fixed $Q_{s_1}, \cdots, Q_{s_n}$ there are $\binom{n - (s_1 + \cdots + s_n)}{r - (s_1 + \cdots + s_n)}$ of r-subsets I_r containing all these subsets.

Now, the conditions (*1) and (*2) are symmetric with respect to $s_1, \cdots, s_n$, so that the summation $\sum^*$ above is a symmetric function of $s_1, \cdots, s_n$. If some $s_i = 1$, then the contribution in $\sum^*$ of the summand is 0. Therefore, assume that for a given sequence $s_1, \cdots, s_n$ of integers, exactly a_2 of them are equal to 2, exactly a_3 of them are equal to 3, $\cdots$, and exactly a_k of them are equal to k, and put

$$t = 2a_2 + 3a_3 + \cdots + ka_k \,,$$
$$u = a_2 + a_3 + \cdots + a_k \,.$$

Then the corresponding summand in (9) is

$$(-1)^{t-u}1^{a_2}2^{a_3} \cdots (k-1)^{a_k}\binom{n-t}{r-t}k^{r-t} \,.$$

13.—Now let π be a non-unitary partition

$$\pi\colon \begin{cases} t = 2a_2 + 3a_3 + \cdots + ka_k \,, \\ u = a_2 + a_3 + \cdots + a_k \,, \end{cases}$$

then we define *structural constants* $J(\pi)$ of L as follows. First choose a set of $u = a_2 + a_3 + \cdots + a_k$ distinct integers, say

$$p_{2,1}, \cdots, p_{2,a_2}, p_{3,1}, \cdots, p_{3,a_3}, \cdots, p_{k,1}, \cdots, p_{k,a_k}$$

from $\{1, \cdots, n\}$. Next for each integer $p_{2,i}$ choose a set of two columns of L each containing the integer $p_{2,i}$; similarly for each integer $p_{3,i}$ choose a set of three columns of L each containing the integer $p_{3,i}$; $\cdots$, and for each integer $p_{k,i}$ choose a set of k columns of L each containing the integer $p_{k,i}$, in such a way that the

$$t = 2a_2 + 3a_3 + \cdots + ka_k$$

columns of L selected are all distinct. The number of the choices above is, by definition, $J(\pi)$.

14.—Thus we have that

$$\alpha_r = \sum_{\pi} (-1)^{t-u}1^{a_2}2^{a_3} \cdots (k-1)^{a_k}\binom{n-t}{r-t} k^{r-t}J(\pi) \,. \tag{10}$$

Returning to § 8, (7) we have

$$\begin{aligned} N &= \sum_{\pi} (-1)^{t-u}1^{a_2}2^{a_3} \cdots (k-1)^{a_k} J(\pi) \sum_{r=t}^{n} (-1)^r\binom{n-t}{r-t}k^{r-t}(n-r)\,! \\ &= \sum_{\pi} (-1)^{u}1^{a_2}2^{a_3} \cdots (k-1)^{a_k}J(\pi)(n-t)! \sum_{r=t}^{n} (-1)^{r-t}\frac{k^{r-t}}{(r-t)!} \\ &= \sum_{\pi} (-1)^{u}1^{a_2}2^{a_3} \cdots (k-1)^{a_k}J(\pi)S_{n-t} \,, \end{aligned} \tag{11}$$

where

$$S_n = n! \sum_{r=0}^{n} \frac{(-k)^r}{r!} = \sum_{r=0}^{n}\binom{n}{r}(-k)^r(n-r)! = (E-k)^n 0\,! \tag{12}$$

with the shift operator E applied on the function $x!$.

15.—We are now in a position to define the *structure polynomial* $f(x)$ of the Latin rectangle L. This is by definition

$$f(x) = \sum_{\pi} (-1)^u 1^{a_2} 2^{a_3} \cdots (k-1)^{a_k} J(\pi) x^{n-t} ,$$

where the summation is over all non-unitary partitions

$$\pi: \begin{cases} t = 2a_2 + 3a_3 + \cdots + ka_k , \\ u = a_2 + a_3 + \cdots + a_k . \end{cases}$$

Take $r = n$ in (10). Then α_n, the number of SDR's of the indexed set $\{C(1), \cdots, C(n)\}$, which is nothing but the number N^* of permutations *imbedded in* the Latin rectangle L, is written as

$$N^* = (-1)^n f(-k) .$$

More generally we have

$$\alpha_r = \frac{(-1)^r}{(n-r)!} f^{(n-r)}(-k) ,$$

and also

$$N = f(E - k)0! ,$$

by using (10) and (11).

16.—THEOREM 1. *Let N denote the number of permutations discordant with a given $k \times n$ Latin rectangle L, and N^* the number of permutations imbedded in L. Then we have*

$$N = \int_0^\infty e^{-x} f(x - k) dx ,$$
$$N^* = (-1)^n f(-k) ,$$

for the structure polynomial $f(x)$ of L.

PROOF. In view of the formulas given in § 15, we have only to remember that

$$E^n 0! = n! = \int_0^\infty e^{-x} x^n dx .$$

17.—Before proceeding to an asymptotic study of the number N we have to make some simple evaluations of the structural constants $J(\pi)$ of L. In fact some of these constants are independent of the individual Latin rectangles. For instance, $J(0) = 1$, where 0 is the partition π with $t = u = 0$. Similarly we have that

$$J(m) = n \binom{k}{m} , \tag{13}$$

where m stands for the partition π with $t = m, u = 1$. The lowest structural constant, as measured by t, is $J(2, 2)$, where $2, 2$ represents the partition with $t = 4, u = 2$. For this constant we have

$$J(2, 2) = \binom{n}{2}\binom{k}{2}^2 - n\binom{k}{2}(k-1)^2 + \square \,, \tag{14}$$

where $\square$ is the number of the 2×2 subrectangles of the form $\begin{smallmatrix} a & a \\ b & b \end{smallmatrix}$ contained in the *Hall rectangle* (the rectangle arising from a Latin rectangle by ignoring the order of appearance of integers in each column) of L.

The above relation (14) is proved by making use of the sieve process, applied on the system of two integers a, b, two columns containing a, and two columns containing b. The condition needed to count for $J(2, 2)$ is that the four columns involved are distinct. More precisely let Ω be the set of all systems $(a, b, C_1, C_2, D_1, D_2)$, were a and b are distinct integers taken out of $\{1, \cdots, n\}$, C_1 and C_2 are columns of L each containing the integer a, and D_1 and D_2 are the columns of L containing the integer b. Let x_{ij} denote the subset of Ω consisting of the systems $(a, b, C_1, C_2, D_1, D_2)$ such that $C_i = D_j$, for $i, j = 1, 2$. Then we have that

$$8J(2, 2) = \#\prod_{i=1}^{2}\prod_{j=1}^{2}(\Omega - x_{ij}) \,,$$

and for the symbolic subset that

$$\prod_{i=1}^{2}\prod_{j=1}^{2}(\Omega - x_{ij}) = \Omega - \sum_{i=1}^{2}\sum_{j=1}^{2} x_{ij} + x_{11}x_{22} + x_{12}x_{21} \,,$$

since $x_{i_1 j_1}x_{i_2 j_2} = \emptyset$ unless $i_1 \neq i_2$ and $j_1 \neq j_2$. We obtain (14) by

$$\#\Omega = (n)_2(k)_2^2, \qquad \#x_{ij} = n(k)_2(k-1)^2 \,,$$

and

$$\#x_{11}x_{22} = \#x_{12}x_{21} = 4\square \,.$$

By the well-known property of the sieve process that the partial sums are in excess and defect, alternately, of the true value, we see from (14) that

$$\binom{n}{2}\binom{k}{2}^2 \geqq J(2, 2) \geqq \binom{n}{2}\binom{k}{2}^2 - n\binom{k}{2}(k-1)^2 \,. \tag{15}$$

18.—We need the following inequality for a general structural constant $J(\pi)$.

$$J(\pi) \leqq \frac{(n)_u}{a_2! a_3! \cdots a_k!}\binom{k}{2}^{a_2}\binom{k}{3}^{a_3}\cdots\binom{k}{k}^{a_k} . \tag{16}$$

In fact, by the definition of $J(\pi)$ in §13, we see that the number of choices of u-subsets of $\{1, \cdots, n\}$ subdivided in the pattern described in §13 is $(n)_u/(a_2!a_3! \cdots a_k!)$, and the number of choices of t columns not necessarily all distinct is $\binom{k}{2}^{a_2}\binom{k}{3}^{a_3}\cdots\binom{k}{k}^{a_k}$ if the u integers are already chosen. The true value $J(\pi)$ may be obtained by applying the sieve process similar to that for $J(2, 2)$ in §17, but we shall content ourselves with the crude inequality (16).

III. The Asymptotic Number of Latin Rectangles.

19.—With all the preparations above we now proceed to obtain an asymptotic formula for N. It follows from Theorem 1 that

$$\begin{aligned} N &= P + R \, , \\ P &= \int_k^\infty e^{-x} f(x-k)dx = e^{-k}\int_0^\infty e^{-x}f(x)dx = e^{-k}f(E)0! \, , \\ R &= \int_0^k e^{-x}f(x-k)dx \, . \end{aligned}$$

We first observe that the "remainder term" R is relatively small. Namely we have

$$|R| < (2k)^n \, .$$

Indeed we have by using (16) that

$$\begin{aligned} |R| &\leqq \int_0^k e^{-x}\sum_\pi 1^{a_2}2^{a_3}\cdots(k-1)^{a_k}J(\pi)k^{n-t}dx \\ &\leqq \sum_\pi (n)_u \frac{1}{a_2!}\left(\frac{1}{2!}\right)^{a_2}\frac{1}{a_3!}\left(\frac{2}{3!}\right)^{a_3}\cdots\frac{1}{a_k!}\left(\frac{k-1}{k!}\right)^{a_k}(k)_2^{a_2}(k)_3^{a_3}\cdots(k)_k^{a_k}k^{n-t} \\ &\leqq k^n\sum_{u=0}^{[n/2]}(n)_u \sum_{a_2+a_3+\cdots+a_k=u}\frac{1}{a_2!}\left(\frac{1}{2!}\right)^{a_2}\frac{1}{a_3!}\left(\frac{2}{3!}\right)^{a_3}\cdots\frac{1}{a_k!}\left(\frac{k-1}{k!}\right)^{a_k} \\ &\leqq k^n\sum_{u=0}^{[n/2]}(n)_u\left\{\text{coefficient of } x^u \text{ in } \exp\left(\frac{1}{2!}x+\frac{2}{3!}x+\cdots\right)\right\} \\ &= k^n\sum_{u=0}^{[n/2]}(n)_u\frac{1}{u!} = k^n\sum_{u=0}^{[n/2]}\binom{n}{u} < 2^nk^n \, . \end{aligned}$$

In the above transformation we have used that

$$\frac{1}{2!}+\frac{2}{3!}+\cdots = \sum_{r=2}^\infty\frac{r-1}{r!} = \sum_{r=2}^\infty\frac{1}{(r-1)!} - \sum_{r=2}^\infty\frac{1}{r!} = 1 \, . \tag{17}$$

20.—As a matter of fact we have that $N \sim e^{-k}n!$ as $n \to \infty$ if k is relatively small. For that purpose we first prove

$$\left|\frac{R}{e^{-k}n!}\right| < (c_1 k n^{-1})^n , \qquad c_1 \text{ absolute constant} . \tag{18}$$

In fact from the inequality in § 19 for R we have

$$\left|\frac{e^k R}{n!}\right| < \frac{e^n 2^n k^n}{n!} = \frac{(2e)^n k^n}{e^{-n} n^{n+\frac{1}{2}} \sqrt{2\pi}}\left(1 + O\left(\frac{1}{n}\right)\right)$$
$$\leqq (2e^2)^n (kn^{-1})^n \left(1 + O\left(\frac{1}{n}\right)\right) .$$

21.—Next consider the "principal term" $P = e^{-k} f(E)0!$ of N. We have

$$e^k P = f(E)0! = \sum_{\pi} (-1)^u 1^{a_2} 2^{a_3} \cdots (k-1)^{a_k} J(\pi)(n-t)! .$$

Denote the partial sums of the above for $t \leqq 4$ and for $t \geqq 5$ by T_1 and T_2 respectively. Then we have from (13) and (15) that

$$T_1 = n! - \binom{k}{2} n(n-2)! - 2\binom{k}{3} n(n-3)!$$
$$- 3\binom{k}{4} n(n-4)! + \binom{n}{2}\binom{k}{2}^2 (n-4)! + O(k^4 n^{-3})(n-4)!$$
$$= n!\left(1 - \frac{\binom{k}{2}}{n-1} - \frac{2\binom{k}{3}}{(n-1)_2} + \frac{\binom{k}{2}^2}{2(n-1)_2} + O(k^4 n^{-3})\right) .$$

22.—For the partial sum T_2 we have

$$| T_2 | \leqq \sum_{\pi \text{ with } t>5} 1^{a_2} 2^{a_3} \cdots (k-1)^{a_k} J(\pi)(n-t)!$$
$$\leqq n! \sum_{\pi \text{ with } t\geqq 5} \frac{(n-t)!}{(n-u)!} \frac{1}{a_2!}\binom{k}{2}^{a_2} \frac{1}{a_3!}\binom{k}{3}^{a_3} \cdots \frac{1}{a_k!}\binom{k}{k}^{a_k} .$$

Here, in view of $2u \leqq t$, we notice the following inequality

$$\frac{(n-t)!}{(n-u)!} < (c_1 n^{-\frac{1}{2}})t . \qquad c_1 \text{ absolute constant} . \tag{19}$$

In fact, for $t = n$ the inequality follows from $n - u \geqq n/2$ by using Stirling's formula. For $t < n$ we have

$$\frac{(n-t)!}{(n-u)!} \leqq \frac{(n-t)!}{\Gamma\left(n - \frac{t}{2} + 1\right)}$$
$$= \frac{e^{-n+t}(n-t)^{n-t+1/2}}{e^{-n+t/2}\left(n - \frac{t}{2}\right)^{n-t/2+1/2}}\left(1 + O\left(\frac{1}{n-t}\right)\right)$$
$$\leqq e^{t/2}\left(n - \frac{t}{2}\right)^{-t/2}\left(1 + O\left(\frac{1}{n-t}\right)\right)$$

$$\leqq e^{t/2}\left(\frac{n}{2}\right)^{-t/2}\left(1+O\left(\frac{1}{n-t}\right)\right)$$
$$= (2e)^{t/2}n^{-t/2}\left(1+O\left(\frac{1}{n-t}\right)\right).$$

Thus (19) is verified, and returning to T_2 we have that

$$\left|\frac{T_2}{n!}\right| \leqq \sum_{\pi \text{ with } t\geqq 5} c_1 n^{-t/2}\frac{1}{a_2!}\left(\frac{1}{2!}\right)^{a_2}\frac{1}{a_3!}\left(\frac{1}{3!}\right)^{a_3}\cdots$$
$$< c_3(kn^{-\frac{1}{2}})^5,$$

with an absolute constant c_3, by using the relation (17).

23.—Combining the results of §§ 20–22 we have

$$\frac{e^k N}{n!} = 1 - \frac{\binom{k}{2}}{n-1} - \frac{2\binom{k}{3}}{(n-1)_2} + \frac{\binom{k}{2}^2}{2(n-1)_2} + $$
$$+ O((c_1 k n^{-1})^n) + O(k^4 n^{-3}) + O(k^5 n^{-5/2})$$
$$= 1 - \frac{\binom{k}{2}}{n-1} - \frac{2\binom{k}{3}}{(n-1)_2} + \frac{\binom{k}{2}^2}{2(n-1)_2} + O(k^5 n^{-5/2})$$
$$= \exp\left(-\frac{\binom{k}{2}}{n-1} - \frac{2\binom{k}{3}}{(n-1)_2} + O(k^5 n^{-5/2})\right).$$

We have proved

THEOREM 2. *The number N of permutations discordant with a given $k \times n$ Latin rectangle satisfies*

$$N = n!\,\exp\left(-k - \frac{\binom{k}{2}}{n-1} - \frac{2\binom{k}{3}}{(n-1)_2} + O(k^5 n^{-5/2})\right)$$

for $k < n^{\frac{1}{2}-\varepsilon}$, ε a positive constant.

24.—Write l in place of k in the above Theorem and take the product over $l = 0, 1, \cdots, k-1$. Then we obtain the following asymtotic relation for the number $f(k, n)$ of $k \times n$ Latin rectangles.

THEOREM 3. *The number $f(k, n)$ of $k \times n$ Latin rectangles satisfies the relation*

$$f(k, n) = (n!)^k \exp\left(-\binom{k}{2} - \frac{\binom{k}{3}}{n-1} - \frac{2\binom{k}{4}}{(n-1)_2} + O(k^6 n^{-5/2})\right)$$

for $k < n^{5/12-\varepsilon}$, ε *a positive constant.*

Notice that $f(k, n) \sim e^{-k(k-1)/2}(n!)^k$ if $k < n^{\frac{1}{3}-\varepsilon}$, ε a positive constant.

25.—Theorem 2 and 3 imply the following relations.

$$N = e^{-k}n!\left(1 - \frac{\binom{k}{2}}{n} + \frac{\binom{k}{2}(3k^2 - 11k + 4)}{12n^2} + O(k^4 n^{-2-\varepsilon})\right)$$

for $k < n^{\frac{1}{2}-\varepsilon}$,

$$f(k, n) = e^{-k(k-1)/2}(n!)^k\left(1 - \frac{\binom{k}{3}}{n} + \frac{\binom{k}{3}\left(\binom{k}{3} - k + 1\right)}{2n^2} + O(k^6 n^{-2-\varepsilon})\right)$$

for $k < n^{\frac{1}{3}-\varepsilon}$.

It is to be pointed out that the coefficient of n^{-2} in the last formula is erraneously stated in a paper of Erdös and Kaplansky [1].

26.—If we assume for the moment that the approximation given in Theorem 3 is still valid for $k = n$, then we would have the following approximation for the number $L(n)$ of reduced Latin squares of order n,

$$n!(n-1)!L(n) \sim (n!)^n \exp\left(-\frac{n(9n-13)}{12}\right),$$

and the accompanying table gives the true and the approximated for $L(n)$.

n	$L(n)$	apprroximahon
4	4	1
5	56	14
6	9,048	2,016
7	16,942,080	4,901,429
8	535,281,401,856	284,422,209,900
9	unknown	5.291333×10^{17}
10	unknown	4.082019×10^{25}

The value $L(8)$ is taken from [6].

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